

Qualifying Exam for Ph.D. Candidacy
Department of Physics
Fall 2022
Part I

Instructions:

- The following problems are intended to probe your understanding of basic physical principles. When answering each question, indicate the principles being applied and any approximations required to arrive at your solution. *If information you need is not given, you may define a variable or make a reasonable physical estimate, as appropriate.* Your solutions will be evaluated based on clarity of physical reasoning, clarity of presentation, and accuracy.
- Please use a new blue book for each question. Remember to write your name and the problem number on the cover of each book.
- We suggest you read all *four* of the problems before beginning to work them. You should reserve time to attempt every problem.

Fundamental constants, conversions, etc.:

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Planck's constant	h	$6.626 \times 10^{-34} \text{ J s}$
	$\hbar = h/2\pi$	$1.055 \times 10^{-34} \text{ J s}$
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Permittivity constant	ϵ_0	$8.854 \times 10^{-12} \text{ F m}^{-1}$
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Standard atmospheric pressure	1 atmosphere	$1.01 \times 10^5 \text{ N m}^{-2}$
Stefan-Boltzmann constant	σ	$5.67 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}$
Electron rest mass	m_e	$9.109 \times 10^{-31} \text{ kg} = 0.5110 \text{ MeV } c^{-2}$
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Definite integrals:

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}. \quad \int_0^\infty x^n e^{-x} dx = \Gamma(n+1) = n!. \quad (\text{I-1})$$

Indefinite integrals:

$$\int \frac{x}{(x^2 + a^2)^n} dx = \frac{1}{2(1-n)} \frac{1}{(x^2 + a^2)^{n-1}} n + c \quad \text{for } n \neq 0, 1 \quad (\text{I-2})$$

Gradient in spherical polar coordinates (r, θ, ϕ) and cylindrical coordinates (r, φ, z) :

$$\vec{\nabla} = \vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \vec{e}_\phi \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi}, \quad \vec{\nabla} = \vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\varphi \frac{1}{r} \frac{\partial}{\partial \varphi} + \vec{e}_z \frac{\partial}{\partial z}, \quad (\text{I-3})$$

Laplacian in spherical polar coordinates (r, θ, ϕ) :

$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2}. \quad (\text{I-4})$$

Laplacian in cylindrical coordinates (r, θ, z) :

$$\nabla^2 f = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}. \quad (\text{I-5})$$

Series expansions:

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n \quad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad (\text{I-6})$$

Legendre polynomials

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n_0}^{\infty} t^n P_n(x) \quad P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n \quad (\text{I-7})$$

$$\int_{-1}^1 dx P_m(x) P_n(x) = \frac{1}{2n+1} \delta_{mn} \quad (\text{I-8})$$

I-1. A surprisingly accurate approximation for the motion of a mass m orbiting a black hole of mass M can be obtained by using ordinary **non-relativistic Newtonian mechanics**, but slightly modifying the usual $1/r$ Keplerian potential to

$$U(r) = -\frac{GmM}{r - r_g}.$$

Here G is the gravitational constant and r_g is the radius of the black-hole event horizon. Orbits with $r < r_g$ are inside the black hole and so unphysical.

a) By means of a Lagrangian or otherwise, obtain, for a general potential $U(r)$ and orbital angular momentum l , an equation for the radius $r(t)$ in the form

$$m\ddot{r} + \frac{\partial W_{\text{eff}}(r, l)}{\partial r} = 0 .$$

You must give an explicit equation for W_{eff} .

b) Use the potential $U(r)$ given above to find the value of l that will allow a circular orbit of radius r_0 around the black hole.

c) Explain how you would use some property of W_{eff} to determine whether the circular orbit you found in part (b) is stable or unstable when the particle is given a small kick that does not alter its orbital angular momentum.

d) Exploit your result from part (b) to show that for the potential $U(r)$ no circular orbit in the range $r_g < r_0 < 3r_g$ is stable.

I-2. Neutrino oscillation occurs because neutrino mass eigenstates are not the same as neutrino flavor eigenstates. When a neutrino is created it is always in a flavor eigenstate — meaning that it is an electron, muon or tau neutrino.

In this problem we will consider only oscillation between the electron neutrino ν_e and the muon neutrino ν_μ .

We will call the mass eigenstates $|\nu_1\rangle$ for mass m_1 and $|\nu_2\rangle$ for mass m_2 . The two flavor eigenstates are related to the mass eigenstates via a mixing angle θ as

$$\begin{aligned} |\nu_e\rangle &= \cos\theta|\nu_1\rangle + \sin\theta|\nu_2\rangle \\ |\nu_\mu\rangle &= -\sin\theta|\nu_1\rangle + \cos\theta|\nu_2\rangle . \end{aligned}$$

When an electron neutrino is created in a source, its state $|\psi\rangle$ will be a linear superposition of the two mass eigenstates with masses m_1 and m_2 . Depending on how it is created, these mass eigenstates may have different energies E_1 , E_2 and corresponding momenta p_1 , p_2 .

a) What are the amplitudes $\langle x=0, t=0, \nu_1|\psi\rangle$ and $\langle x=0, t=0, \nu_2|\psi\rangle$ for the electron neutrino to be in each of the m_1 and m_2 mass eigenstates immediately after it is created?

b) The neutrino is detected some time t later and a large distance x from the source. At this distance the space-time part of the wavefunction can be taken to be a unit-amplitude plane wave. What are the amplitudes $\langle x, t, \nu_1|\psi\rangle$ and $\langle x, t, \nu_2|\psi\rangle$?

c) Suppose that $E_1 = E_2$. What is the difference between the momenta p_1 and p_2 ? (The mass difference is very small so the approximation $\sqrt{1+x^2} \simeq 1+x^2/2$ can be used where appropriate).

d) With the $E_1 = E_2$ assumption, what is the phase difference $\Delta\phi$ at the detector between the two amplitudes $\langle x, t, \nu_1 | \psi \rangle$ and $\langle x, t, \nu_2 | \psi \rangle$? Express your answer in terms of m_1, m_2, E, \hbar , the speed of light c and the distance x .

e) Use your answer to part (d) to determine the probability $P_e(x)$ that the detected neutrino remains an electron neutrino, and the probability $P_\mu(x)$ that the neutrino is detected as a muon neutrino.

I-3. A neutron star is essentially a degenerate Fermi gas. Consider the situation where all the neutrons in the star are ultrarelativistic, i.e. they have energy $\epsilon = pc$ where the momentum is $p = \hbar k$ and k is the wavenumber. Consider a large cube of volume V filled with N of these ultrarelativistic neutrons in thermal equilibrium at temperature T .

a) Treat the neutrons as quantum particles in the cube with periodic boundary conditions. The energy density of states $D(\epsilon)$ is defined by replacing the sum over states by an integral according to

$$\sum_{\text{states}} (...) \rightarrow \int (...) D(\epsilon) d\epsilon ,$$

where (...) represents any physical quantity of interest. Show that

$$D(\epsilon) = A\epsilon^\alpha ,$$

where you should determine the quantities A and α . (Do not forget that the neutron has spin 1/2).

b) Determine the Fermi energy ϵ_F (i.e. the chemical potential at temperature $T = 0$) in terms of the neutron number density $n = N/V$ and other physical constants.

c) Write down an integral expression for the average energy density $u = U/V$ of the system at temperature T , but do not attempt to simplify it. Explain why, in the limit $k_B T \ll \epsilon_F$ the expression for $u(T = 0)$ is a good approximation for $u(T)$.

d) Evaluate $u(T = 0)$ exactly, writing your final answer in terms of n and ϵ_F .

I-4. The figure shows an ideal dipole of dipole moment p that lies in the center of a grounded conducting spherical shell of inner radius R_1 and outer radius R_2 .

a) Obtain a fully explicit expression for $V(r, \theta)$ in the three regions $r \leq R_1$, $R_1 < r < R_2$, and $r > R_2$.

b) Compute the surface charge density $\sigma(\theta)$ on the inner surface of the grounded conductor.

c) Compute the surface charge density $\sigma(\theta)$ on the outer surface of the grounded conductor

d) Now suppose that in addition to the dipole a point charge q is placed in the center of the sphere. Repeat your calculation of the surface charge distributions on the inner and outer surfaces of the conducting shell.

Hint: Recall that the general axisymmetric solution to Laplace's equation can be written

$$V(r, \theta) = \sum_{n=0}^{\infty} \left(a_n r^n + \frac{b_n}{r^{n+1}} \right) P_n(\cos \theta)$$

and that $b_1 = p$ for an isolated dipole. See the formula sheet for explicit expressions for the $P_n(\cos \theta)$.

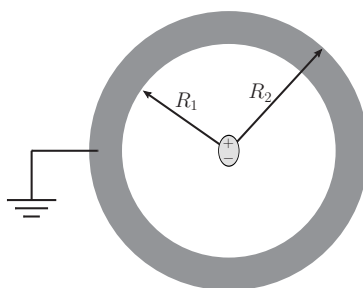


Figure I-4: An ideal dipole that lies in the center of a grounded conducting spherical shell of inner radius R_1 and outer radius R_2 .

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II-1. On the two ends of a massless rod, we put two masses m_1 and m_2 . m_1 is resting on a horizontal frictionless surface and we keep m_2 at a height h . Now we let m_2 fall.

- a) Assuming that the mass m_1 is held fixed, find the velocity of m_2 when it hits the surface.
- b) Assuming that m_1 is allowed to slide, find the velocities v_1 of m_1 and v_2 of m_2 at the time when m_2 hits the surface.
- c) For the setup at point b), find the positions of the two masses when m_2 hits the surface.

II-2. We consider a particle of mass m in one dimension, bound in the potential well $V(r)$ given by

$$V(r) = \begin{cases} +\infty & r \leq 0, \\ -V_0 & 0 < r < r_0, \\ 0 & r \geq r_0. \end{cases} \quad (\text{with } V_0 > 0) \quad (\text{II-9})$$

- a) Show that for $V_0 < \frac{\hbar^2}{2m} \left(\frac{\pi}{2r_0} \right)^2$, this potential has no bound states and that, at the threshold $V_0 = \frac{\hbar^2}{2m} \left(\frac{\pi}{2r_0} \right)^2$, the potential has a single (marginal) bound state which is the energy eigenstate $\psi_0(r)$ of energy $E_0 = 0$.

We consider now two particles of masses m_1 and m_2 in three dimensions, bound by the radial potential $V(r)$ given above with $r = |\vec{x}_1 - \vec{x}_2|$ the distance between the two particles. We assume that the potential is at the threshold for having a single very-weakly-bound state of energy $E_0 = 0$.

- b) Use the result of (a), together with the general structure of the ground-state wave function, $\Phi_0(r, \theta, \phi) = \psi_0(r)/r$, to determine the value of the potential V_0 corresponding to the existence of a single (marginal) bound state of energy $E_0 = 0$. How does V_0 depend on m_1 and m_2 ?

The deuteron is a bound state of a proton and a neutron. We model the strong nuclear force between the two as a spherical well with the potential $V(r)$ given above. The radius r_0 of the potential well is assumed to be given by the Compton wavelength of the pion, $r_0 \approx \hbar/(m_{\pi^0}c) \simeq 1.46$ fm.

- c) Assuming that the deuteron is a very-weakly-bound state of a proton and a neutron, determine the depth V_0 of the potential well (expressed in MeV). Check the consistency of this approximation against the observation that a gamma ray of energy $E_\gamma \simeq 2.23$ MeV breaks the deuteron into a proton and a neutron.

II-3. We are given a container of gas and are required to determine the equation of state of that gas. Lab measurements of the isothermal compressibility $\kappa_T = -\frac{1}{V} \left(\frac{\partial V}{\partial P} \right)_T$ and the coefficient of isobaric thermal expansion $\alpha_P = \frac{1}{V} \left(\frac{\partial V}{\partial T} \right)_P$ are

$$\kappa_T = \frac{T}{v} f(P) \quad \alpha_P = \frac{R}{vP} + \frac{A}{vT^2}$$

where T is the temperature, $v = V/n$ is the molar volume, and $f(P)$ is some function of pressure P that was not determined by lab measurements.

- a) find the function $f(P)$.
b) find the equation of state $V = V(P, T)$ of the gas.

II-4. In a given region of space we have a static magnetic field, which, in a cylindrical reference frame (r, ϕ, z) , is symmetric around the z axis, i.e. is independent of ϕ , and can be written $\mathbf{B} = \mathbf{B}(r, z)$. The field component along z is $B_z(z) = B_0 z/L$, where B_0 and L are constant parameters.

a) Find the radial component B_r close to the z axis.

A particle of magnetic polarizability α (such that it acquires an induced magnetic dipole moment $\mathbf{m} = \alpha \mathbf{B}$ in a magnetic field \mathbf{B}), is located close to the z axis.

b) Find the potential energy of the particle in the magnetic field.

c) Assuming $\alpha < 0$, find equilibrium position(s) for the particle, and find the frequency of oscillations for small displacements from equilibrium either along z or r (let M be the mass of the particle).